

THE AFFINE WEYL GROUP SYMMETRY OF DESARGUES MAPS AND OF THE NON-COMMUTATIVE HIROTA-MIWA SYSTEM

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ABSTRACT. We study recently introduced Desargues maps, which provide simple geometric interpretation of the non-commutative Hirota–Miwa system. We characterize them as maps of the A -type root lattice into a projective space such that images of vertices of any basic regular N -simplex are collinear. Such a characterization is manifestly invariant with respect to the corresponding affine Weyl group action, which leads to related symmetries of the Hirota–Miwa system.

1. INTRODUCTION

The Desargues maps, as defined in [14], are maps $\phi : \mathbb{Z}^N \rightarrow \mathbb{P}^M(\mathbb{D})$ of multidimensional integer lattice into projective space of dimension $M \geq 2$ over a division ring \mathbb{D} , such that for any pair of indices $i \neq j$ the points $\phi(n)$, $\phi(n + \varepsilon_i)$ and $\phi(n + \varepsilon_j)$ are collinear; here $\varepsilon_i = (0, \dots, \overset{i}{1}, \dots, 0)$ is the i -th element of the canonical basis of \mathbb{R}^N . Under mild genericity conditions, by an appropriate choice of homogeneous coordinates, such maps are described [14] in terms of solutions $\phi : \mathbb{Z}^N \rightarrow \mathbb{D}_*^{M+1}$ of the linear system (we consider the right vector spaces over \mathbb{D})

$$(1.1) \quad \phi(n + \varepsilon_i) - \phi(n + \varepsilon_j) = \phi(n)U_{ij}(n), \quad i \neq j \leq N,$$

with the corresponding functions $U_{ij} : \mathbb{Z}^N \rightarrow \mathbb{D}_*$.

The linear system (1.1) is well known in soliton theory [11, 40]. Its compatibility condition is the following nonlinear system

$$(1.2) \quad U_{ij}(n) + U_{ji}(n) = 0, \quad U_{ij}(n) + U_{jk}(n) + U_{ki}(n) = 0,$$

$$(1.3) \quad U_{kj}(n)U_{ki}(n + \varepsilon_j) = U_{ki}(n)U_{kj}(n + \varepsilon_i),$$

for distinct triples i, j, k , called the non-commutative Hirota–Miwa system [35, 37, 40]. Equation (1.3) allows to introduce the potentials $\rho_i : \mathbb{Z}^N \rightarrow \mathbb{D}_*$ such that

$$(1.4) \quad U_{ij}(n) = [\rho_i(n)]^{-1} \rho_i(n + \varepsilon_j).$$

When \mathbb{D} is commutative, i.e. a field, the functions ρ_i can be parametrized in terms of a single potential τ (the tau-function)

$$(1.5) \quad \rho_i(n) = (-1)^{\sum_{k>i} n_k} \frac{\tau(n + \varepsilon_i)}{\tau(n)}.$$

Then equations (1.2) can be rewritten as the Hirota–Miwa [22, 31] system (called also the discrete Kadomtsev–Petviashvili (KP) system)

$$(1.6) \quad \tau(n + \varepsilon_i)\tau(n + \varepsilon_j + \varepsilon_k) - \tau(n + \varepsilon_j)\tau(n + \varepsilon_i + \varepsilon_k) + \tau(n + \varepsilon_k)\tau(n + \varepsilon_i + \varepsilon_j) = 0, \quad i < j < k,$$

whose fundamental role in soliton theory is described, for example, in [31, 26, 3].

As it was shown in [14] the four dimensional compatibility of the Desargues maps is equivalent to the celebrated Desargues theorem [2] of the incidence geometry. However, the above definition of the Desargues maps does not exhibit the well known symmetry of the Desargues configuration [29]; see also discussion in [14]. In this paper we propose more geometric definition of the Desargues maps, where instead of the \mathbb{Z}^N lattice we use the root lattice $Q(A_N)$. This point of view allows to see, from the very beginning, the

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corresponding affine Weyl group $W(A_N)$ symmetry of the Desargues maps and of the non-commutative Hirota–Miwa system.

At this point we should mention a recent related work [47] of W. K. Schief, who found a symmetric description of the Laplace sequence of two dimensional quadrilateral lattices [46, 12] in terms of certain maps of the three dimensional lattice of face-centered cubic combinatorics, which is the root lattice $Q(A_3)$ [48]; see also Section 3.2 for more details. We remark that in the integrable discrete geometry the face-centered cubic closest sphere packing lattice has been used in [16, 13] in a different context of a sub-lattice version of Miwa’s discrete BKP equation [31]. The idea of checking compatibility, on three dimensional lattices generated by tetrahedra, of a system of linear equations on triangular lattices has been announced to me also by M. Nieszporski [34] in the context of a six-point linear problem [33], which contains as reductions both the four point and the three point linear problems.

We remark that the A_{K-1} root lattice served in [18, 17] as the parametrization space of the Laplace transformations of K -dimensional quadrilateral lattices [15]. In this context it is present explicitly in [14], where it was shown that the theory of K -dimensional quadrilateral lattices and their Laplace transformations is the same as the theory of $(2K - 1)$ -dimensional Desargues maps. In describing this equivalence it was convenient to introduce implicitly the A_{2K-1} root lattice, but the full geometric flavor of this change of variables has not been observed there.

Finally, it is worth to mention that our research has been also motivated by (extended) affine Weyl groups symmetries of discrete Painlevé systems [42, 44, 41, 27]. The geometric picture associated to the theory of Painlevé equations is connected usually with the representation theory of the affine Weyl groups in terms of birational actions on rational surfaces [45]. It is well known [1, 36, 38, 20, 27] that the Painlevé type equations can be obtained as symmetry reductions of soliton systems. As the present work brings to the light the affine Weyl group symmetry already on the level of the Hirota–Miwa system, it can be considered as a prologue to the incidence geometric description of the integrability of the Painlevé type equations.

Notice that, by multidimensional compatibility of the Desargues maps, the dimension N of the root lattice can be arbitrarily large. Therefore our paper provides geometric explanation of the appearance of the A_∞ root lattice in the Kadomtsev–Petviashvili hierarchy, which is encoded [31] in the Hirota–Miwa system, other then the standard one [10, 25] via the theory of representations of the Lie algebra $\mathfrak{gl}(\infty)$.

The construction of the paper is as follows. We collect first in Section 2 some useful facts on the A_N root lattice and the corresponding affine Weyl group. Then in Section 3 we present the A_N root lattice description of the Desargues maps and their corresponding affine Weyl group symmetry. Finally, in Section 4 we study the action of the A_N affine Weyl group on solutions of the non-commutative Hirota–Miwa system.

We remark that we do not discuss here nor the Darboux type transformations of the Desargues maps nor the corresponding Bäcklund type transformations of the Hirota–Miwa system. These results are known on the algebraic level [39, 19], and their incidence geometry interpretation follows from the known geometric meaning [17] of the corresponding transformations of the quadrilateral lattices. Because also here, like in many integrable discrete systems [30], there is no essential difference between the Darboux transformation and a step into new dimension of the lattice system, the permutability of the transformations follows from the multidimensional compatibility of the Desargues maps. The details are of no particular interest in the context of this paper and will be presented elsewhere.

2. THE A_N ROOT LATTICE AND ITS AFFINE WEYL GROUP

In this Section we recall necessary facts on the A_N root lattice, its Delaunay tiles and the corresponding affine Weyl group action. The subject is fairly standard, see for example [4, 9, 6, 24, 32]. Notice that in order to simplify the presentation we adjusted some general formulas of the root lattices theory to this specific case.

2.1. The A_N root lattice. The $N \geq 2$ -dimensional root lattice $Q(A_N)$, where the terminology comes from theory of simple Lie algebras [4], is generated by vectors along the edges of regular N -simplex. If we take the vertices of the simplex to be the vectors of the canonical (and orthonormal with respect to the

standard scalar product) basis in \mathbb{R}^{N+1}

$$\mathbf{e}_i = (0, \dots, \overset{i}{1}, \dots, 0), \quad 1 \leq i \leq N+1,$$

then the generators are

$$(2.1) \quad \varepsilon_j^i = \mathbf{e}_i - \mathbf{e}_j, \quad 1 \leq i \neq j \leq N+1,$$

which identifies the lattice as the set of all vectors $(m_1, \dots, m_{N+1}) \in \mathbb{Z}^{N+1}$ of integer coordinates with zero sum $m_1 + \dots + m_{N+1} = 0$. We consider the $Q(A_N)$ lattice as embedded in the N -dimensional vector space

$$\mathbb{V} = \{(x_1, \dots, x_{N+1}) \in \mathbb{R}^{N+1} \mid x_1 + \dots + x_{N+1} = 0\}$$

with scalar product $(\cdot | \cdot)$ inherited from the ambient \mathbb{R}^{N+1} . The standard basis of the root lattice, useful from the point of view the action of the affine Weyl group (see Section 2.2), consists of the so called *simple roots*

$$\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \quad 1 \leq i \leq N.$$

Define the *fundamental weights* $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_N \in \mathbb{V}$ as the dual to the simple root basis

$$(\boldsymbol{\omega}_i | \boldsymbol{\alpha}_j) = \delta_{ij}.$$

Vertices of the weight lattice $P(A_N)$,

$$P(A_N) = \sum_{i=1}^N \mathbb{Z} \boldsymbol{\omega}_i,$$

are the points of the root lattice $Q(A_N)$ and their translates by the fundamental weights.

2.2. The A_N Weyl groups. The *Weyl group* $W_0(A_N)$ is the Coxeter group generated by reflections r_i , $1 \leq i \leq N$, with respect of the hyperplanes through the origin and orthogonal to the corresponding simple roots

$$(2.2) \quad r_i : \mathbf{v} \mapsto \mathbf{v} - 2 \frac{(\mathbf{v} | \boldsymbol{\alpha}_i)}{(\boldsymbol{\alpha}_i | \boldsymbol{\alpha}_i)} \boldsymbol{\alpha}_i.$$

The group $W_0(A_N)$ is isomorphic to the symmetric group S_{N+1} which act permuting the vectors \mathbf{e}_i , $1 \leq i \leq N+1$; the generators r_i are identified then with the transpositions $\sigma_i = (i, i+1)$.

Denote by $\tilde{\boldsymbol{\alpha}}$ the highest root

$$\tilde{\boldsymbol{\alpha}} = -\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_1 + \dots + \boldsymbol{\alpha}_N = \mathbf{e}_1 - \mathbf{e}_{N+1}.$$

The *affine Weyl group* $W(A_N)$ is the Coxeter group generated by r_1, r_2, \dots, r_N and by an additional affine reflection r_0

$$(2.3) \quad r_0 : \mathbf{v} \mapsto \mathbf{v} - \left(1 - 2 \frac{(\mathbf{v} | \tilde{\boldsymbol{\alpha}})}{(\tilde{\boldsymbol{\alpha}} | \tilde{\boldsymbol{\alpha}})}\right) \tilde{\boldsymbol{\alpha}}.$$

In more abstract terms the affine Weyl group $W(A_N)$ is defined by the generators r_0, r_1, \dots, r_N and the relations

$$r_i^2 = 1, \quad (r_i r_j)^2 = 1 \quad (j \neq i, i \pm 1), \quad (r_i r_j)^3 = 1 \quad (j = i \pm 1),$$

where indices are considered modulo $N+1$.

From (2.2)-(2.3) we can obtain the following formulas, which we will need in Section 4.2:

$$(2.4) \quad r_i(n + \varepsilon_k^j) = r_i(n) + \varepsilon_{\sigma_i(k)}^{\sigma_i(j)}, \quad n \in Q(A_N),$$

where $\sigma_0 = (1, N+1)$. In particular

$$(2.5) \quad r_i(n + \boldsymbol{\alpha}_j) = r_i(n) + \boldsymbol{\alpha}_j - a_{ji} \boldsymbol{\alpha}_i, \quad 0 \leq i, j \leq N, \quad n \in Q(A_N),$$

where

$$a_{ij} = (\boldsymbol{\alpha}_i | \boldsymbol{\alpha}_j) = \begin{cases} 2 & i = j, \\ -1 & j = i \pm 1 \pmod{N+1}, \\ 0 & \text{otherwise,} \end{cases}$$

is the Cartan matrix of the affine Weyl group $W(A_N)$.

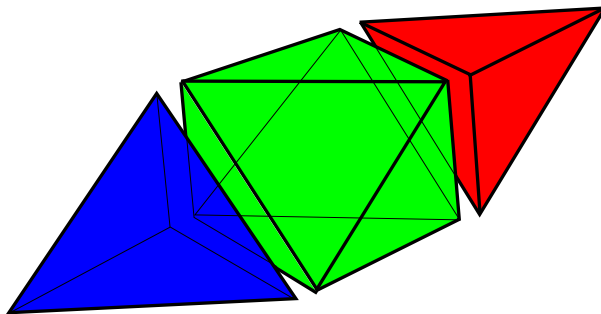


FIGURE 1. The fundamental parallelogram of the A_3 root lattice decomposed into its Delaunay tiles: two tetrahedra $P(1,3)$ and $P(3,3)$, and the octahedron $P(2,3)$.

The convex hull of the vertices $\{0, \omega_1, \dots, \omega_N\}$ is the *fundamental region* of for the action of $W(A_N)$ on \mathbb{V} . The affine Weyl group is the semidirect product

$$(2.6) \quad W(A_N) = Q(A_N) \rtimes W_0(A_N)$$

of the Weyl group by the translations along the root lattice with the action given by (2.2).

2.3. The Delaunay polytopes of the root lattice. The *holes* in the lattice are the points of \mathbb{V} that are locally maximally distant from the lattice. The convex hull of the lattice points closest to a hole is called the *Delaunay polytope*. The Delaunay polytopes of the root lattice $Q(A_N)$ form a tessellation of \mathbb{V} into N convex polytopes $P(k, N)$, $k = 1, \dots, N$, called *ambo-simplices* in [7] or regular *hypersimplices* in [21]. We remark that the tiles $P(1, N)$, congruent to the initial N -simplex of Section 2.1, are of particular interest in our paper. They will be called basic regular N -simplices of the lattice.

The points $Q(A_N) + \omega_k$ are centers of Delaunay tiles congruent to $P(k, N)$. Up to an appropriate affine transformation, which sends the fundamental parallelogram of the A_N root lattice into the standard unit N -hypercube $I_N = [0, 1]^N$, the tile $P(k, N)$ can be identified with the region (slice) of I_N between two hyperplanes (see Figure 1)

$$\hat{P}(k, N) = \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid 0 \leq x_1, \dots, x_N \leq 1, k-1 \leq x_1 + \dots + x_N \leq k\}.$$

The hypersimplex $P(k, N)$ can be equivalently (up to an affine transformation) described as the section of the hypercube I_{N+1} by the corresponding hyperplane

$$\tilde{P}(k, N) = \{(x_1, \dots, x_{N+1}) \in \mathbb{R}^{N+1} \mid 0 \leq x_1, \dots, x_{N+1} \leq 1, x_1 + \dots + x_{N+1} = k\},$$

where the projection $(x_1, \dots, x_N, x_{N+1}) \mapsto (x_1, \dots, x_N)$ sends $\tilde{P}(k, N)$ to $\hat{P}(k, N)$. From the last description it follows that the 1-skeleton of $P(k, N)$ is the so called Johnson graph $J(N+1, k)$: its vertices are labelled by k -point subsets of $\{1, 2, \dots, N+1\}$, and edges are the pairs of such sets with $(k-1)$ -point intersection.

The following result [6, 32] is of the fundamental importance for our paper.

Lemma 2.1. *The affine Weyl group acts on the Delaunay tiling by permuting tiles within each class $P(k, N)$.*

Remark. The ambo-simplices $P(k, N)$ and $P(N-k+1, N)$ can be identified by applying a point reflection symmetry, which is not an element of the affine Weyl group. For our purposes it is important to keep the difference.

In [32] one can find a detailed description of affine Weyl group orbits of facets of various dimension of the Delaunay tiles of root lattices.

Corollary 2.2. *The following results are either explicitly stated in [32] or can be without difficulty derived using the method presented there:*

- (1) *With respect to the $W(A_N)$ -action the K -dimensional facets of Delaunay tiles of $Q(A_N)$ are exactly K ambo-simplices $P(k, K)$, $1 \leq k \leq K$, where the notation comes from $Q(A_K)$ sublattices.*

- (2) No two different regular N -simplices $P(1, N)$ in $Q(A_N)$ share more than a 0-dimensional facet, i.e. a vertex of the root lattice; in any vertex there meet exactly $N + 1$ such simplices. All the 2-facets of $P(1, N)$ are the $P(1, 2)$ equilateral triangles in the number of $\binom{N+1}{3}$.
- (3) The K -facet $P(K - 1, K)$ has exactly:
- (i) $\binom{K+1}{2}$ vertices;
 - (ii) $(K + 1) \binom{K}{2}$ 1-facets (edges);
 - (iii) $\binom{K+1}{3}$ 2-facets $P(1, 2)$, of which exactly $K - 1$ meet in any vertex of the tile and no two share an edge, and $(K + 1) \binom{K}{3}$ 2-facets $P(2, 2)$;
 - (iv) $\binom{K+1}{4}$ 3-facets $P(2, 3)$, which are regular octahedra.

3. THE ROOT LATTICE DESCRIPTION OF THE DESARGUES MAPS

In this Section we study geometrically the Desargues maps and their affine Weyl group symmetry. Its appearance is natural once their definition is restated in terms of the root lattice. We also discuss the resulting symmetry of the generalized Desargues configurations which are responsible for multidimensional compatibility of the maps. We postpone for the next Section the study of the corresponding properties of the non-commutative Hirota–Miwa system.

3.1. The Desargues maps of the root lattice. Consider the \mathbb{Z}^N coordinates in the root lattice $Q(A_N)$ by the following identification

$$(3.1) \quad \mathbb{Z}^N = \sum_{i=1}^N \mathbb{Z} \epsilon_i^{N+1} = Q(A_N).$$

Then the Desargues map condition [14], recalled at the beginning of the Introduction, can be formulated as collinearity of images of points labelled by $n, n + \epsilon_i^{N+1}$, $i = 1, \dots, N$. Because those are the vertices of a basic N -simplex $P(1, N)$ of the root lattice, then one arises to more geometric characterization of Desargues maps, which can eventually be taken as their definition.

Proposition 3.1. *Under the above identification (3.1) of $Q(A_N)$ with \mathbb{Z}^N the Desargues maps are the maps $\phi : Q(A_N) \rightarrow \mathbb{P}^M$ such that the vertices of each basic N -simplex $P(1, N)$ are mapped into collinear points.*

Remark. The image of a fundamental parallelootope of the root lattice under the Desargues map was called in [14] a Desargues cube.

The above proposition/definition from the very beginning exhibits the affine Weyl group symmetry of Desargues maps, what can be immediately inferred from Lemma 2.1.

Theorem 3.2. *If $\phi : Q(A_N) \rightarrow \mathbb{P}^M$ is a Desargues map then also for any element w of the affine Weyl group $W(A_N)$ the map $\phi \circ w$ is a Desargues map, where we consider the natural action of $W(A_N)$ on the root lattice $Q(A_N)$.*

Corollary 3.3. *Because the theory of K -dimensional quadrilateral lattices and their Laplace transformations is equivalent [14] to the theory of $(2K - 1)$ -dimensional Desargues maps then the $W(A_{2K-1})$ affine Weyl group symmetry applies, with appropriate modifications, to quadrilateral lattices as well.*

3.2. Generalized Desargues configurations. Consider the image of a K dimensional facet of type $P(K - 1, K)$ under generic Desargues map. The analysis will be made on the basis of results presented in Corollary 2.2. First notice that for $K = 3$ we obtain the Veblen configuration [2], which is responsible for the three-dimensional compatibility of the Desargues maps [14]: six vertices of the octahedron $P(2, 3)$ are mapped into six coplanar points, while its four facets $P(1, 2)$ (the other four are $P(2, 2)$) are mapped into four lines.

Remark. In [47] the maps of the three-dimensional lattice of face centered cubic combinatorics, i.e. the $Q(A_3)$ root lattice [48], into \mathbb{R}^3 with the property that vertices of the (bipartite) octahedra are mapped into points of the Veblen configuration (named there the Menelaus configuration, because in the affine context it is related to the Menelaus theorem) have been called the Laplace–Darboux lattices. Moreover, it was shown there that such maps provide a symmetric description of the Laplace sequences of two-dimensional quadrilateral lattices [46, 12].

By point (3) of Corollary 2.2 the Desargues map images of $\binom{K+1}{2}$ vertices of $P(K-1, K)$ and of its $\binom{K+1}{3}$ 2-facets $P(1, 2)$ give a configurations of points and lines, respectively, which satisfies the following conditions:

- (1) every line is incident with exactly three points,
- (2) every point is incident with exactly $K-1$ lines,
- (3) it contains exactly $\binom{K+1}{4}$ Veblen configurations.

In the case of $K=4$ we obtain the Desargues configuration, which is responsible for the four-dimensional compatibility of the Desargues maps [14]. For $K>4$ we obtain generalizations of the Desargues configuration, called binomial configurations and studied in [23, 29, 43].

Let us discuss the symmetry of the generalized Desargues configurations. By properties of the Johnson graphs of the ambo-simplices $P(K-1, K)$ the points of the configuration are labelled by $(K-1)$ -point subsets of $\{1, 2, \dots, K+1\}$, while the lines of the configuration are labelled by the $(K-2)$ -point subsets. A point is incident with a line if the line labels are contained in the point labels. The symmetry group of the K -th generalized Desargues configuration is thus the symmetric group of $(K+1)$ elements $S_{K+1} = W_0(A_K)$; we remark that we do not take into account the point-line duality in the special case $K=4$ of the original Desargues configuration *on the plane*.

Remark. There is a simple geometric description of the origin of the symmetry of generalized Desargues configurations, attributed by Coxeter [8] to Cayley [5] in the basic case $K=4$. Combinatorially, instead of $(K-1)$ -point subsets of $\{1, 2, \dots, K+1\}$ to label the points of the configuration one uses the complementary 2-point subsets (and 3-point subsets to label the lines).

Given $K+1$ points in general position in the projective space \mathbb{P}^K , consider lines joining pairs of the points, and planes through the triplets. Intersected by a generic hyperplane in \mathbb{P}^K the lines and planes give K -th generalized Desargues configuration of points and lines on the hyperplane. The symmetry group permutes the $K+1$ points.

4. THE AFFINE WEYL GROUP SYMMETRY OF THE HIROTA–MIWA SYSTEM

Below we transfer geometric considerations of the previous Section into the language of the Hirota–Miwa system and of its symmetries. We remark, that similarly one can study the affine Weyl group symmetry of the non-commutative discrete modified KP system and the generalized lattice spin system [37] (called also the non-commutative Schwarzian discrete KP system [28]), because they are gauge equivalent [14] to the Hirota–Miwa system.

4.1. \mathbb{Z}^N sectors of a Desargues map. Fix \mathbb{Z}^N -coordinates in the A_N root lattice by the identification (3.1), and consider the linear problem of the Desargues map (1.1) in the Hirota–Miwa gauge adjusted to this choice of basis

$$(4.1) \quad \phi^{N+1}(n + \varepsilon_i^{N+1}) - \phi^{N+1}(n + \varepsilon_j^{N+1}) = \phi^{N+1}(n) U_{ij}^{N+1}(n), \quad 1 \leq i \neq j \leq N,$$

with the corresponding potentials ρ_i^{N+1} , $1 = 1, \dots, N$, such that

$$U_{ij}^{N+1}(n) = [\rho_i^{N+1}(n)]^{-1} \rho_j^{N+1}(n + \varepsilon_j^{N+1}).$$

In choosing the above coordinates we used one of the $N+1$ tiles $P(1, N)$ meeting in the point $n \in Q(A_N)$. Another choice of a "sector", i.e. a basis $\{\varepsilon_j^i\}$, where the index i is fixed, and $j \neq i$, along edges of another such a tile, should give a similar linear problem. The following result shows the gauge and the potentials adjusted to such an equivalent choice.

Lemma 4.1. *The functions $\phi^i : Q(A_N) \rightarrow \mathbb{D}_*^{M+1}$ given by*

$$(4.2) \quad \phi^i(n) = (-1)^{(n|\epsilon_i^{N+1})} \phi^{N+1}(n) [\rho_i^{N+1}(n)]^{-1},$$

satisfy the linear system in the i -th sector

$$(4.3) \quad \phi^i(n + \epsilon_j^i) - \phi^i(n + \epsilon_k^i) = \phi^i(n) U_{jk}^i(n), \quad i, j, k \text{ distinct},$$

where

$$(4.4) \quad U_{jk}^i(n) = [\rho_j^i(n)]^{-1} \rho_j^i(n + \epsilon_k^i),$$

and the potentials ρ_j^i are given by

$$(4.5) \quad \rho_j^i(n) = \begin{cases} \rho_j^{N+1}(n) [\rho_i^{N+1}(n)]^{-1}, & j \neq N+1, \\ [\rho_i^{N+1}(n)]^{-1}, & j = N+1. \end{cases}$$

Proof. To obtain (4.3) with $j = N+1$ it is enough to apply definitions (4.2) and (4.5) to equation (4.1). Then in order to check the case $k = N+1$ one has to demonstrate that $U_{N+1,j}^i(n) = -U_{j,N+1}^i(n)$, which follows from the basic relation $U_{ij}^{N+1}(n) = -U_{ji}^{N+1}(n)$ expressed in terms of the potentials $\rho_i^{N+1}(n)$ and $\rho_j^{N+1}(n)$. One uses also the following simple rules to express vectors of the new basis in terms of the initial vectors

$$\epsilon_j^i = \epsilon_j^{N+1} - \epsilon_i^{N+1}, \quad \text{where by definition } \epsilon_k^k = \mathbf{0}.$$

Finally, to prove equation (4.3) with $j, k \neq N+1$ one needs to use two equations of the linear problem (4.1) for the pairs i, j and i, k , and the equation $U_{ij}^{N+1}(n) + U_{ki}^{N+1}(n) = -U_{jk}^{N+1}(n)$ of the system (1.2). \square

Corollary 4.2. *Formulas (4.2) and (4.5) are self-consistent, i.e. in the place of the index $N+1$ one can take an arbitrary index, i.e.*

$$\phi^i(n) = (-1)^{(n|\epsilon_i^j)} \phi^j(n) [\rho_i^j(n)]^{-1}.$$

In particular one can check that

$$(4.6) \quad \rho_j^i(n) \rho_i^k(n) = \rho_j^k(n), \quad \text{where by definition } \rho_i^i = 1.$$

From Lemma 4.1 we get the following conclusions:

Corollary 4.3. *The functions U_{ij}^ℓ satisfy the system (1.2)-(1.3)*

$$(4.7) \quad U_{ij}^\ell(n) + U_{ji}^\ell(n) = 0, \quad U_{ij}^\ell(n) + U_{jk}^\ell(n) + U_{ki}^\ell(n) = 0,$$

$$(4.8) \quad U_{kj}^\ell(n) U_{ki}^\ell(n + \epsilon_j^\ell) = U_{ki}^\ell(n) U_{kj}^\ell(n + \epsilon_i^\ell),$$

while the potentials ρ_j^i satisfy corresponding equations

$$(4.9) \quad \rho_\ell^i(n) \rho_i^\ell(n + \epsilon_j^\ell) + \rho_\ell^j(n) \rho_j^\ell(n + \epsilon_i^\ell) = 0,$$

$$(4.10) \quad \rho_\ell^i(n) \rho_i^\ell(n + \epsilon_j^\ell) + \rho_\ell^j(n) \rho_j^\ell(n + \epsilon_k^\ell) + \rho_\ell^k(n) \rho_k^\ell(n + \epsilon_i^\ell) = 0.$$

Notice that changing the sector can be understood as a symmetry transformation of both the linear and nonlinear systems. Other generators of symmetries are translations and permutations of indices within a fixed sector.

4.2. The affine Weyl group action on the edge potentials. Let $E(A_N)$ denote the set of *oriented* edges of the root lattice $Q(A_N)$, i.e. elements of $E(A_N)$ are ordered pairs $[n, n + \epsilon_j^i]$, where $n \in Q(A_N)$. Define the function $\rho : E(A_N) \rightarrow \mathbb{D}$ by

$$(4.11) \quad \rho([n, n + \epsilon_j^i]) = \rho_j^i(n).$$

It is convenient to distinguish the simple root functions $\rho^i = \rho_{i+1}^i$, $i = 1, \dots, N$, which are attached to the simple roots directions. One can check, using the condition (4.6), that for $i < j$ we have

$$(4.12) \quad \rho_j^i = \rho^{j-1} \dots \rho^i, \quad \rho_i^j = (\rho_j^i)^{-1}.$$

Let us define also the function ρ^0 as attached to the direction of the root α_0 , which by (4.12) gives

$$(4.13) \quad \rho^0 = (\rho^N \rho^{N-1} \dots \rho^1)^{-1}.$$

Define the action of the affine Weyl group on the functions ρ_j^i through its action on the oriented edges of the root lattice, i.e.

$$(4.14) \quad (w.\rho)([n, n + \varepsilon_j^i]) = \rho(w^{-1}[n, n + \varepsilon_j^i]).$$

Proposition 4.4. *The action of the generators r_i , $i = 0, \dots, N$, of the affine Weyl group on the functions ρ_k^j , is given by*

$$(4.15) \quad (r_i.\rho_k^j)(n) = \rho_{\sigma_i(k)}^{\sigma_i(j)}(r_i(n)),$$

where σ 's are the transpositions $\sigma_i = (i, i+1)$, $i = 1, \dots, N$, and $\sigma_0 = (1, N+1)$.

Proof. The conclusion follows from equations (2.4), (4.14), and from involutivity of the generators r_i . \square

By equations (2.5) and (4.12) we have:

Corollary 4.5. *The action of the generators r_i , $i = 0, \dots, N$, of the affine Weyl group on the functions ρ^j , $j = 0, \dots, N$, is given by*

$$(4.16) \quad (r_i.\rho^j)(n) = [(\rho^i)^{-a_{ji}^U} \rho^j (\rho^i)^{-a_{ji}^L}](r_i(n)),$$

where a_{ji}^U and a_{ji}^L are the "upper" and the "lower" parts of the Cartan matrix of the affine Weyl group $W(A_N)$

$$(4.17) \quad a_{ij}^L = \begin{bmatrix} 1 & 0 & & -1 \\ -1 & 1 & 0 & \\ & -1 & 1 & \ddots \\ & & \ddots & \ddots & 0 \\ 0 & & & -1 & 1 \end{bmatrix}, \quad a_{ij}^U = \begin{bmatrix} 1 & -1 & & 0 \\ 0 & 1 & -1 & \\ & 0 & 1 & \ddots \\ & & \ddots & \ddots & -1 \\ -1 & & & 0 & 1 \end{bmatrix}.$$

The counterpart of Theorem 3.2 on the level of the non-commutative Hirota–Miwa system can be then stated as follows.

Proposition 4.6. *Transformations of the potentials ρ_j^i given in equation (4.15) generate the affine Weyl group $W(A_N)$ symmetry of solutions of the non-commutative Hirota–Miwa system (4.9)–(4.10).*

Proof. By Lemma 4.1 and Proposition 4.4 we infer that the action of the generators of the affine Weyl group on the edge potentials is consistent with the action on the wave functions ϕ^j given by

$$(4.18) \quad (r_i.\phi^j)(n) = \phi^{\sigma_i(j)}(r_i(n)).$$

\square

4.3. The τ -functions. Among the edge potentials ρ_j^i only N of them enter nontrivially into the nonlinear system (4.9)–(4.10). They can be chosen by fixing a sector, i.e. fixing the index i . Another choice is given by the simple root potentials ρ^i , $i = 1, \dots, N$. We give below another natural set of such basic potentials (one of them will be redundant).

Notice that condition (4.6) allows to introduce functions $\tau_i : Q(A_N) \rightarrow \mathbb{D}$, $i = 1, \dots, N+1$, such that

$$(4.19) \quad \rho_j^i(n) = \tau_j(n) [\tau_i(n)]^{-1}.$$

Equations (4.9)–(4.10) rewritten in terms of the τ -functions read

$$(4.20) \quad [\tau_i(n)]^{-1} \tau_i(n + \varepsilon_j^\ell) [\tau_\ell(n + \varepsilon_j^\ell)]^{-1} + [\tau_j(n)]^{-1} \tau_j(n + \varepsilon_i^\ell) [\tau_\ell(n + \varepsilon_i^\ell)]^{-1} = 0,$$

$$(4.21) \quad [\tau_i(n)]^{-1} \tau_i(n + \varepsilon_j^\ell) [\tau_\ell(n + \varepsilon_j^\ell)]^{-1} + [\tau_j(n)]^{-1} \tau_j(n + \varepsilon_k^\ell) [\tau_\ell(n + \varepsilon_k^\ell)]^{-1} + [\tau_k(n)]^{-1} \tau_k(n + \varepsilon_i^\ell) [\tau_\ell(n + \varepsilon_i^\ell)]^{-1} = 0.$$

It is not difficult to verify the following result.

Proposition 4.7. *The action of the affine Weyl group on the edge potentials ρ_j^i follows from the action on the τ -functions given by*

$$(4.22) \quad (r_i.\tau_j)(n) = \tau_{\sigma_i(j)}(r_i(n)).$$

Remark. Notice that by equation (4.2) the function

$$(4.23) \quad (-1)^{(n|\mathbf{e}_i)} \phi^i(n) \tau_i(n), \quad i = 1, \dots, N+1,$$

is independent of the index i .

Remark. For \mathbb{D} commutative, i.e. a field, one can resolve equations (4.20) by expressing N τ -functions in terms of one of them, for example

$$(4.24) \quad \tau_i(n) = (-1)^{\sum_{\ell > i} n_\ell^{N+1}} \tau_{N+1}(n + \varepsilon_i^{N+1}), \quad n = \sum_{\ell=1}^N n_\ell^{N+1} \varepsilon_\ell^{N+1}, \quad i \neq N+1,$$

compare with equation (1.5). Then the remaining equations (4.21) reduce to the standard form (1.6) of the Hirota-Miwa system.

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